流體力學

※ 參考書 ※

- ✯ Yih : Fluid Mechanics
- ✯ Yuan : Fundation of Fluid Mechanics
- ✯ Cole : Fluid Dynamics
- ✯ Batchelor : An Introduction to Fluid Mechanics
- ✯ Landau : Fluid Mechanics
- ✯ Via : Vector to Cartesian Tensor (藍色書皮)

詹志正 筆記 1983年 原稿
蔡偉雄 講授 1980年 真傳

流力的範圍：討論流體的運動與平衡的現象。

與流力有關的Field：天文物理、生物學、生物物理、海洋學、電磁流力(Magnetic-hydrodynamic)、電漿物理、氣象、醫藥-------

什麼是流體？ ⟶ 理想的流體不能抵抗切應力，在彈性限度內，固體可抵抗切應力。

☆☆☆ 向量分析 ☆☆☆

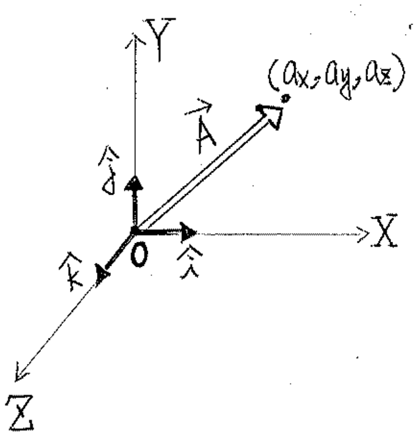
Chapter 1 Vector and Cartesian Tensor

§1-1 Representation of vector

$\hat{A}$ or $\vec{A}$

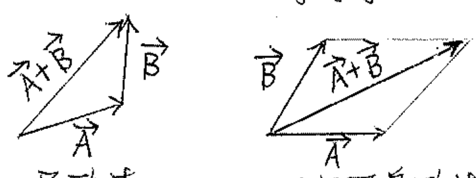
$\vec{A}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}=(a_x,a_y,a_z)$

同理，$\vec{B}=b_x\hat{i}+b_y\hat{j}+b_z\hat{k}$



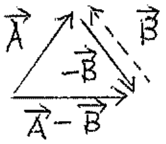
(1) 加減法

$\vec{A}\pm\vec{B}=(a_x\pm b_x)\vec{i}+(a_y\pm b_y)\vec{j}+(a_z\pm b_z)\vec{k}$



三角形法　　平行四邊形法

$\vec{A}=\vec{B} \Longleftrightarrow a_x=b_x$，$a_y=b_y$，$a_z=b_z$



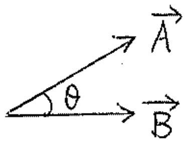
又 $-\vec{B}=-b_x\hat{i}-b_y\hat{j}-b_z\hat{k}$，$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

力矩是向量，功是純量

乃因 力矩＝力×力臂 且 功＝力×位移

(2) 純量積(scalar product)

$$\vec{A}\cdot\vec{B}=AB\cos\theta$$

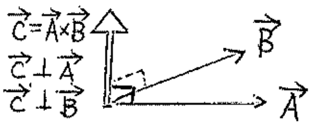


$$\Rightarrow \begin{array}{l}\hat{i}\cdot\hat{i}=1,\ \hat{j}\cdot\hat{j}=1,\ \hat{k}\cdot\hat{k}=1\\ \hat{i}\cdot\hat{j}=0,\ \hat{j}\cdot\hat{k}=0,\ \hat{k}\cdot\hat{i}=0\end{array}$$

$$\vec{A}\cdot\vec{B}=(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})\cdot(b_x\hat{i}+b_y\hat{j}+b_z\hat{k})=a_xb_x+a_yb_y+a_zb_z$$

(3) 向量積(vector product)

$$\vec{C}=\vec{A}\times\vec{B}$$



$|\vec{C}|=C=AB\sin\theta\ (0\le\theta<\pi)$

$\vec{C}$之方向：用右手定則決定之，它垂直$\vec{A}$與$\vec{B}$

$\hat{i}\times\hat{j}=\hat{k}$，$\hat{j}\times\hat{k}=\hat{i}$，$\hat{k}\times\hat{i}=\hat{j}$，$\vec{B}\times\vec{A}=-\vec{C}$

$$\vec{C}=(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})\times(b_x\hat{i}+b_y\hat{j}+b_z\hat{k})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$$

(4) 二項積(dyadic product)

$$\overleftrightarrow{C}=(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})(b_x\hat{i}+b_y\hat{j}+b_z\hat{k})$$

$$=\begin{array}{l}a_xb_x\hat{i}\hat{i}+a_xb_y\hat{i}\hat{j}+a_xb_z\hat{i}\hat{k}\\a_yb_x\hat{j}\hat{i}+a_yb_y\hat{j}\hat{j}+a_yb_z\hat{j}\hat{k}\\a_zb_x\hat{k}\hat{i}+a_zb_y\hat{k}\hat{j}+a_zb_z\hat{k}\hat{k}\end{array}=\begin{bmatrix}a_xb_x&a_xb_y&a_xb_z\\a_yb_x&a_yb_y&a_yb_z\\a_zb_x&a_zb_y&a_zb_z\end{bmatrix}$$

§ 1-2 Scalar and vector field

scalar field：某物理純量為空間各點之函數：$\phi(x,y,z)$

vector field：某物理向量為空間各點之向量函數：

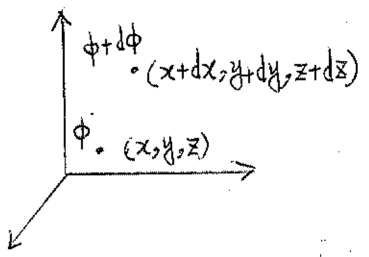
$$\vec{F}(x,y,z)=F_x(x,y,z)\hat{i}+F_y(x,y,z)\hat{j}+F_z(x,y,z)\hat{k}$$

[Ex] $\vec{V}(x,y,z)=V_x(x,y,z)\hat{i}+V_y(x,y,z)\hat{j}+V_z(x,y,z)\hat{k}$ ※

§ 1-3 Gradient

del operator：$\vec{\nabla}=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

$\vec{\nabla}\phi$：gradient of $\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$



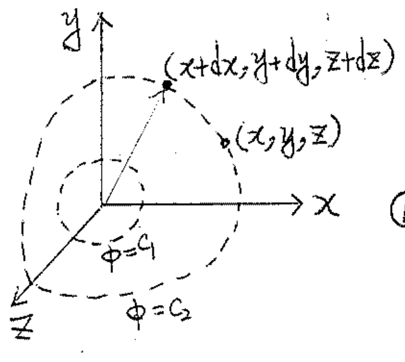
$\phi=\phi(x,y,z)$, $d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

$\Rightarrow d\phi=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}dx+\hat{j}dy+\hat{k}dz\right)$

$\Rightarrow d\phi=\vec{\nabla}\phi\cdot d\vec{r}$

$\Rightarrow d\phi=|\vec{\nabla}\phi|\cdot|d\vec{r}|\cdot\cos\theta=|\vec{\nabla}\phi|(dr)\cos\theta$

若 $d\vec{r}\parallel\vec{\nabla}\phi\Rightarrow\theta=0$，$d\phi=|\vec{\nabla}\phi|\,dr$，$|\vec{\nabla}\phi|=\frac{d\phi}{dr}$



$\phi=x^2+y^2+z^2$：ϕ 是 x,y,z 的函數

在 $\phi=c$ 之等值面上：$\therefore\phi=c=x^2+y^2+z^2$

①若 $d\vec{r}$ 在 $\phi=c$ 之等值面上

$\Rightarrow d\phi=0\Rightarrow d\phi=(\vec{\nabla}\phi)\cdot(d\vec{r})=0$ ∴

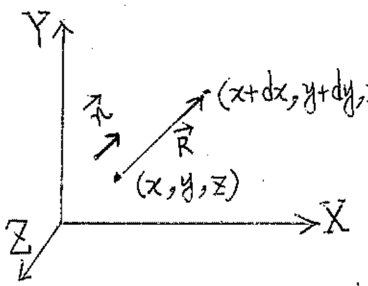
$\vec{\nabla}\phi\perp d\vec{r}\Rightarrow\vec{\nabla}\phi$ 垂直 $\phi=c$ 之等位面

②若 $d\vec{r}$ 在 $\phi=c$ 之法線上

$d\phi=|\vec{\nabla}\phi|\,dr$，$|\vec{\nabla}\phi|=\frac{d\phi}{dr}$

$\phi(x,y,z)$: scalar function

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \quad : \text{vector}$$



$$d\vec{R} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$\vec{n} = \frac{d\vec{R}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} = \text{unit vector}$$

direction derivative : $\dfrac{d\phi}{ds} = \dfrac{\partial\phi}{\partial x}\cdot\dfrac{dx}{ds} + \dfrac{\partial\phi}{\partial y}\cdot\dfrac{dy}{ds} + \dfrac{\partial\phi}{\partial z}\cdot\dfrac{dz}{ds} = \vec{\nabla}\phi\cdot\vec{n}$

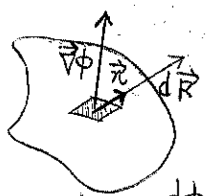
$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

we get the formula $\boxed{\dfrac{d\phi}{ds} = \vec{\nabla}\phi\cdot\vec{n}}$

(1) The component of $\vec{\nabla}\phi$ in any direction gives the rate of change $\frac{d\phi}{ds}$ in that direction : $\frac{d\phi}{ds} = |\vec{\nabla}\phi|\cos\theta$, $\left(\frac{d\phi}{ds}\right)_{max} = |\vec{\nabla}\phi|$, $(\theta = 0)$

(2) $\vec{\nabla}\phi$ points in the direction of max rate of increase of function

(3) The magnitude of $\vec{\nabla}\phi$ equals the max rate of increase of ϕ per unit distance



$\phi = C$ isotimic surface (等值面)

If $\vec{n}$ in the $\phi = C$ isotimic surface $\Rightarrow$ $\dfrac{d\phi}{ds} = 0 = \vec{\nabla}\phi\cdot\vec{n}$

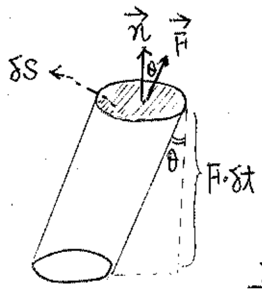
i.e. $\vec{\nabla}\phi \perp \vec{n}$

(4) Through any point (x,y,z) (where $\vec{\nabla}\phi \neq 0$), there passes an isotimic surface $\phi(x,y,z) = C$, $\vec{\nabla}\phi$ is normal to this surface at the point (x,y,z)

§ 1-4 Divergence

$\vec{F} = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k}$ = vector field

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$
$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{a scalar field : divergence of } \vec{F}$$



(註：$\vec{F}$可想為$\vec{v}$《速度》)

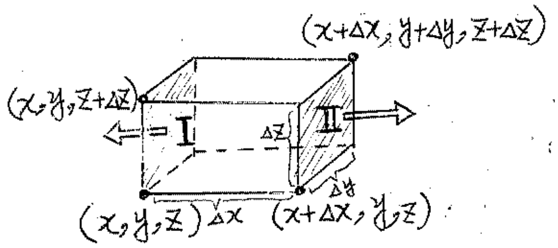
$\delta V = (\delta S)\,F\,\delta t\cos\theta$, 又 $F\cos\theta = \vec{F}\cdot\vec{n}$

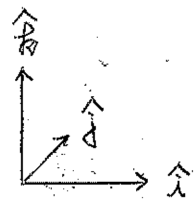
故 $\delta V = \delta t\,\vec{F}\cdot\vec{n}\,\delta S$

$$\frac{\delta V}{\delta t} = \frac{\text{volumn cross area } \delta S}{\text{unit time}} = \vec{F}\cdot\vec{n}\,\delta S$$

= flux of the vector field $\vec{F}$ through the area δS

(where $\vec{n}\,\delta S = \delta\vec{S}$)





area I : $(F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(-\hat{i})\,\Delta z\,\Delta y = -F_x(x,y,z)\,\Delta z\,\Delta y$

area II : $[F_x(x+\Delta x,y,z)\,\hat{i} + F_y\hat{j} + F_z\hat{k}]\cdot(\hat{i})\,\Delta z\,\Delta y = F_x(x+\Delta x,y,z)\,\Delta z\,\Delta y$

then, flux of out area I and II (along x axis) :

$$[F_x(x+\Delta x,y,z) - F_x(x,y,z)]\,\Delta y\,\Delta z = \frac{\partial F_x}{\partial x}\,\Delta x\,\Delta y\,\Delta z$$

similar, $\frac{\partial F_y}{\partial y} \Delta x \Delta y \Delta z$ ---------- along y axis

$\frac{\partial F_z}{\Delta z} \Delta x \Delta y \Delta z$ ---------- along z axis

total flux $= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta x \Delta y \Delta z$

$$\frac{\text{total flux}}{\text{unit volumn}} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla} \cdot \vec{F}$$

Roughly speaking, the divergence of a vector field in a scalar field that tell us, at each point, the extent to which the field diverges from that point.

mass out the region $\Delta x \Delta y \Delta z$ per unit time :

$$\left[\frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} + \frac{\partial (\rho v_z)}{\partial z} \right] \Delta x \Delta y \Delta z = \vec{\nabla} \cdot (\rho \vec{v}) \Delta x \Delta y \Delta z$$

where $\vec{v}$ is the velocity of fluid

but $\frac{\partial m}{\partial t} = \frac{\partial (\rho \Delta x \Delta y \Delta z)}{\partial t}$ ($\because$ conservation of mass)

implies $\vec{\nabla} \cdot (\rho \vec{v}) \Delta x \Delta y \Delta z = - \frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$

hence $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \implies$ equation of continuity

§ 1-5 Curl (or rotation)

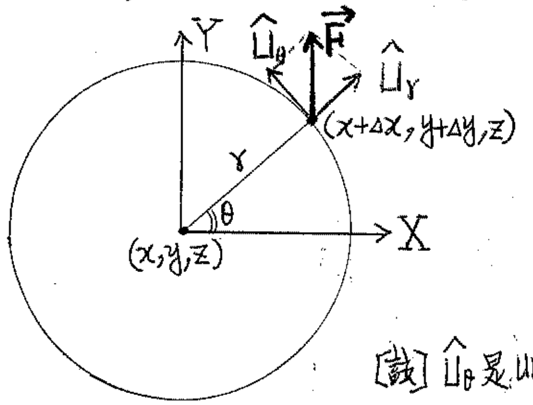
$\vec{F} = F_x(x,y,z)\hat{i} + F_y(x,y,z)\hat{j} + F_z(x,y,z)\hat{k}$: vector field

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad , \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

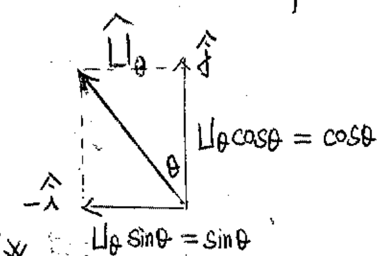
The curl of a vector field is a vector field that gives us at each point, an indication of how the field swirls in the vicinity of that point.
旋渦 鄰近

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \ , \ \vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

$\vec{F}$: velocity of fluid , $\vec{\omega}$: angular velocity of fluid



把 $\hat{U}_\theta$ 放大來分析



[設] $\hat{U}_\theta$ 是 unit vector ※

then $\hat{U}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + 0\,\hat{k}$

so that $\vec{F}\cdot\hat{U}_\theta = -F_x\sin\theta + F_y\cos\theta = -F_x(x+\Delta x, y+\Delta y, z)\sin\theta + F_y(x+\Delta x, y+\Delta y, z)\cos\theta$

$$\text{where}\begin{cases} F_x(x+\Delta x, y+\Delta y, z) = F_x(x,y,z) + \frac{\partial F_x}{\partial x}\Delta x + \frac{\partial F_x}{\partial y}\Delta y \\ \\ F_y(x+\Delta x, y+\Delta y, z) = F_y(x,y,z) + \frac{\partial F_y}{\Delta x}\Delta x + \frac{\partial F_y}{\partial y}\Delta y \end{cases}$$

$$\Rightarrow \vec{F}\cdot\hat{u}_\theta = -\left(F_x+\frac{\partial F_x}{\partial x}r\cos\theta+\frac{\partial F_x}{\partial y}r\sin\theta\right)\sin\theta+\left(F_y+\frac{\partial F_y}{\partial x}r\cos\theta+\frac{\partial F_y}{\partial y}r\sin\theta\right)\cos\theta$$

$$\Rightarrow \overline{\vec{F}\cdot\hat{u}_\theta} = \frac{1}{2\pi}\int_0^{2\pi}\vec{F}\cdot\hat{u}_\theta\,d\theta = \frac{1}{2}r\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right) \Rightarrow \omega_z = \frac{\overline{\vec{F}\cdot\hat{u}_\theta}}{r} = \frac{1}{2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

〈附〉 $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$ 且

$$\int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0$$

及 $\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta = \pi$ ※

similar , $\omega_y = \frac{1}{2}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)$, $\omega_x = \frac{1}{2}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)$, get

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{i}+\frac{1}{2}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{j}+\frac{1}{2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{k} = \frac{1}{2}\vec{\nabla}\times\vec{F}$$

§ 1-6 Laplacian

ϕ : a scalar field , $\vec{F}$: a vector field

$$(\vec{\nabla})\cdot(\vec{\nabla}\phi) = (\vec{\nabla}\cdot\vec{\nabla})\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2\vec{F} = \frac{\partial^2\vec{F}}{\partial x^2}+\frac{\partial^2\vec{F}}{\partial y^2}+\frac{\partial^2\vec{F}}{\partial z^2}$$

→ Laplacian of ϕ (or $\vec{F}$)

§ 1-7 Vector identities

$\vec{\nabla}\cdot(\phi\vec{F}) = \phi\,\vec{\nabla}\cdot\vec{F}+\vec{F}\cdot\vec{\nabla}\phi$, $\vec{\nabla}\times(\phi\vec{F}) = \phi\,\vec{\nabla}\times\vec{F}+\vec{\nabla}\phi\times\vec{F}$

$\boxed{\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\vec{\nabla}\times\vec{F})-\vec{F}\cdot(\vec{\nabla}\times\vec{G})}$, $\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F})-\nabla^2\vec{F}}$

$$\left\{\begin{array}{l}\vec{\nabla}\times(\vec{\nabla}\phi)=0\\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\vec{\nabla}\times\vec{A}=0 \;\Rightarrow\; \vec{A}=\vec{\nabla}\phi\\ \vec{\nabla}\cdot\vec{B}=0 \;\Rightarrow\; \vec{B}=\vec{\nabla}\times\vec{F}\end{array}\right.$$

另外，
$$\begin{aligned}(\vec{A}\cdot\vec{\nabla})\vec{B} = &\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)\hat{i}\\ &+\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right)\hat{j}\\ &+\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)\hat{k}\end{aligned}$$

〔註〕 $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$

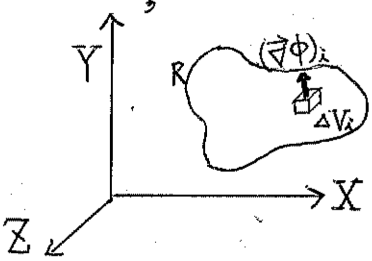
$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

$\vec{A}\cdot\vec{B}\times\vec{C}=\vec{B}\cdot\vec{C}\times\vec{A}=\vec{C}\cdot\vec{A}\times\vec{B}$ ※

§ 1-8 Gradient theorem

V: volumn , σ: surface , l: line , $\vec{n}$: normal unit vector of $d\sigma$

$$\iiint_R \vec{\nabla}\phi\, dV=\iint_S \phi\,\vec{n}\,d\sigma$$



§ 1-9 Divergence theorem

$$\iiint_R \vec{\nabla}\cdot\vec{A}=\iint_S \vec{A}\cdot\vec{n}\,d\sigma \quad 或 \quad \boxed{\iiint_R \vec{\nabla}\cdot\vec{A}\,dV=\iint_S \vec{n}\cdot\vec{A}\,d\sigma}$$

§ 1-10 Stokes theorem

$$\boxed{\iint_S(\vec{\nabla}\times\vec{A})\cdot\vec{n}\,d\sigma=\int_C\vec{A}\cdot\vec{t}\,dl}$$ $\vec{t}$: tangential unit vector of dl

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✲ Chapter 2 The kinematics of fluid flow ✲

§ 2-1 Continuum hypothesis

連續體的意義：把流體的任何物理量當做空間座標及時間座標的連續函數。

（我們把流體當做連續體討論之）

$$\begin{Bmatrix} \text{characteristic length} : L \\ \text{characteristic time} : T \\ \text{characteristic velocity} : V \end{Bmatrix}$$

$L \gg d_0$, d_0 : mean free path of fluid moleculer

〈附〉空氣流動若超過音速，則為可壓縮
空氣流動若不超過音速，則為不可壓縮。※

$$\text{Knudson number} \equiv \frac{d_0}{L} \leq 0.01$$

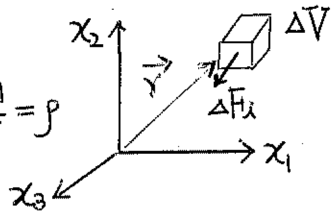
§ 2-2 The idea of stress
應力

Types of acting force $\begin{cases} \text{① body force (超距力)} \\ \text{② surface force (表面力《接觸力》)} \end{cases}$

① body force $\longrightarrow$ The face acting on all elements of the fluid, like gravitan force, electric force etc.

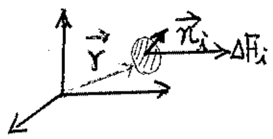
the expression of the body force

$$f_i(\vec{r},t) = \lim_{\Delta V \to 0} \frac{\Delta F_i}{\Delta V} \quad \text{or} \quad \tilde{f}_i(\vec{r},t) = \lim_{\Delta m \to 0} \frac{\Delta F_i}{\Delta m} \quad \therefore f_i = \rho \tilde{f}_i$$



$$\frac{\Delta m}{\Delta V} = \rho \qquad f_i = \lim_{\Delta V \to 0} \frac{\Delta F_i}{\Delta V}$$

② surface force $\longrightarrow$ 表面力是一種接觸力 (contact force)



$$\sigma_i(\vec{r}, \vec{n}_i) = \lim_{\Delta S \to 0} \frac{\Delta F_i}{\Delta S} = \sigma_i : \text{stress vector}$$

surface force are contact forces that act across some surface of the fluid, which may be internal or external, this type of force is usually introduced as a force per unit area at a point in the fluid.

where, ΔF_i : total force that the fluid on the $+n_i$ side extents on the $-n_i$ side, across the surface.

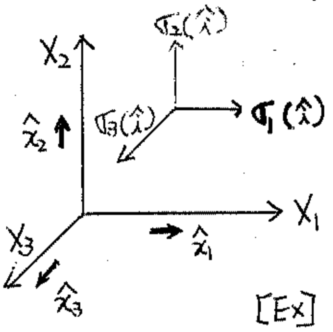
(I) range convention
慣例,定法

$$\vec{A} = a_i = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = (a_1, a_2, a_3)$$

$\hat{i}$: free index, take 1, 2, 3

when $\vec{A} = \vec{B}$ then $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ i.e. $a_i = b_i$

§ 2-3 Notation for stress components



$\sigma_1(\hat{i})$：作用面垂直$\hat{i}$，且作用於$\hat{x}_1$方向的力

同理可知 $\sigma_2(\hat{i})$ 與 $\sigma_3(\hat{i})$ 的意義

而$\sigma_1(\hat{j})$：作用面垂直$\hat{j}$，且作用於$\hat{x}_1$方向的力

推廣而知 $\sigma_2(\hat{j})$，$\sigma_3(\hat{j})$ 以及 $\sigma_1(\hat{k})$，$\sigma_2(\hat{k})$，$\sigma_3(\hat{k})$

[Ex] $\sigma_2(\hat{k})$ 即：作用面垂直$\hat{k}$，且作用於$\hat{x}_2$方向的力 ※

故 $\sigma_{st} = \sigma_t(\hat{s})$ 代表：作用面垂直$\hat{s}$且作用於$\hat{t}$方向的力

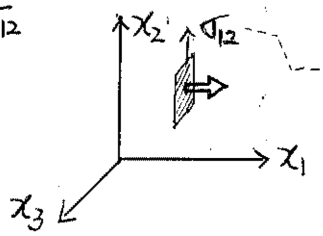
[闡釋] $\sigma_i(\hat{i})$，$\sigma_i(\hat{j})$，$\sigma_i(\hat{k}) = \sigma_{ij}$ = stress tensor

as $\hat{i} = \hat{x}_1$，$\hat{j} = \hat{x}_2$，$\hat{k} = \hat{x}_3$

$$\text{where} \left\{ \begin{array}{lll} \sigma_1(\hat{i}) = \sigma_1(\hat{x}_1) = \sigma_{11}, & \sigma_2(\hat{i}) = \sigma_{12}, & \sigma_3(\hat{i}) = \sigma_{13} \\ \sigma_1(\hat{j}) = \sigma_{21}, & \sigma_2(\hat{j}) = \sigma_{22}, & \sigma_3(\hat{j}) = \sigma_{23} \\ \sigma_1(\hat{k}) = \sigma_{31}, & \sigma_2(\hat{k}) = \sigma_{32}, & \sigma_3(\hat{k}) = \sigma_{33} \end{array} \right\}$$

σ_{ij}：The stress tensor, i.e. the j-th component of the force per unit area exerted on across the surface with normal directed in the x_i axis.

[Ex] σ_{12}



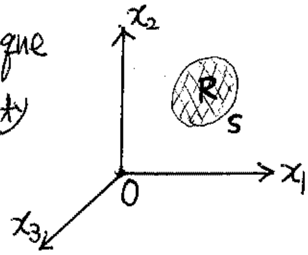
作用面垂直x_1，作用於x_2方向的力

※

§ 2-4 The law of motion

F_i : force , P_i : linear momentum , L_i : torque

H_i : angular momentum , v_i : velocity , ρ : density



$$\dot{P}_i = \frac{dP_i}{dt} = F_i \ , \quad \dot{H}_i = \frac{dH_i}{dt} = L_i$$

[設] $\vec{L} = \vec{r} \times \vec{F}$, $\vec{H} = \vec{r} \times \vec{P}$ ※

$$F_i = \underbrace{\iiint_R f_i(\vec{r}, \overset{\text{time}}{\overset{\downarrow}{t}})\, dV}_{\text{body force}} + \underbrace{\iint_S \sigma_i(\vec{r}, t)\, ds}_{\text{surface force}} \quad ., \text{ as } P_i = \iiint_R \rho v_i \, dV$$

$$\boxed{\frac{d}{dt}\left[\iiint_R \rho v_i \, dV\right] = \iiint_R f_i \, dV + \iint_S \sigma_i \, ds}$$

R是t的函數，dt不可搬進$\iiint_R$內

(固定某一質點的變化率)

$\Uparrow$

$\frac{dF}{dt}$: comoving derivative

$$= \frac{d}{dt}\left[\iiint_R f \, dV\right]$$

$$\neq \iiint_R \frac{df}{dt} \, dV$$

※ 注意：R如果不是t的函數

dt可搬進$\iiint_R$內，舉例如下

(固定某一位置的變化率)

$\Uparrow$

$\left(\frac{\partial F}{\partial t}\right)$ = spatial derivative
空間的

$$= \frac{\partial}{\partial t}\left[\iiint_R f \, dV\right]$$

$$= \iiint_R \frac{\partial f}{\partial t} \, dV$$

$$\vec{L}=\iiint_{R(t)} \vec{r}\times\vec{f}\,dV+\iint_{S(t)} \vec{r}\times\vec{\sigma}\,ds \qquad \text{where } \begin{array}{l}\vec{r}: x_i\\ \vec{f}: f_i\end{array}$$

<i> $\delta_{ij}=\left\{\begin{array}{ll}1 & i=j\\ 0 & i\neq j\end{array}\right\}$ 代表有9個分量

<ii> ϵ_{ijk} 代表有27個分量

$$\epsilon_{ijk}\left\{\begin{array}{ll} =1, & \text{if } i,j,k \text{ is even permutation of } 1,2,3 \quad [\text{Ex}]\ \epsilon_{231}=1 \ ※\\ =-1, & \text{if } i,j,k \text{ is odd permutation of } 1,2,3 \quad [\text{Ex}]\ \epsilon_{132}=-1 \ ※\\ =0, & \text{otherwise} \quad [\text{Ex}]\ \epsilon_{111}=\epsilon_{122}=\epsilon_{113}=\epsilon_{233}=0 \ ※\end{array}\right.$$

由於 $\vec{A}$ 可記為 a_i，$\vec{B}$ 可記為 b_i，因此 $\vec{A}\times\vec{B}$ 可記為 $\epsilon_{ijk}a_j b_k$

(II) summation convention

$\sigma_{ii}=\sigma_{11}+\sigma_{22}+\sigma_{33}=1+1+1=3$ （此時的 i 是 dummy index）

$\epsilon_{ijk}\epsilon_{ijk}=6=\epsilon_{123}\epsilon_{123}+\epsilon_{132}\epsilon_{132}+\epsilon_{213}\epsilon_{213}+\epsilon_{231}\epsilon_{231}+\epsilon_{312}\epsilon_{312}+\epsilon_{321}\epsilon_{321}$

<附> 又 $(\vec{A}\times\vec{B})_1=(a_2b_3-a_3b_2)$ （參考 $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$）

proof: $(\vec{A}\times\vec{B})_1=\epsilon_{1jk}a_jb_k=\epsilon_{123}a_2b_3+\epsilon_{132}a_3b_2=a_2b_3-a_3b_2$ ※

From $L_i=\iiint_{R(t)}\epsilon_{ijk}x_jf_k\,dV+\iint_{S(t)}\epsilon_{ijk}x_j\sigma_k\,ds$

and $H_i=\iiint_{R(t)}\epsilon_{ijk}x_jv_k\rho\,dV$

$$\Rightarrow \frac{d}{dt}\iiint_{R(t)}\epsilon_{ijk}x_jv_k\rho\,dV=\iiint_{R(t)}\epsilon_{ijk}x_jf_k\,dV+\iint_{S(t)}\epsilon_{ijk}x_j\sigma_k\,ds$$

classical notation → index notation

$\vec{\nabla}\phi \longrightarrow \frac{\partial\phi}{\partial x_i} = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right)$

$\vec{\nabla}\cdot\vec{A} \longrightarrow \frac{\partial a_i}{\partial x_i}$

$\vec{\nabla}\times\vec{A} \longrightarrow \epsilon_{ijk}\frac{\partial a_k}{\partial x_j}$

$\vec{A}\times\vec{B} = \epsilon_{ijk}a_j b_k$
$\vec{A}\cdot\vec{B} = a_i b_i$
& $a_{ij}x_j = b_i$ 意即 $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

§ 2-5 Cauchy's formula

$$\left\{\begin{array}{ll} A_{ij} : & \text{covariant tensor} \\ A^{ij} : & \text{contravariant tensor} \\ A^i_j : & \text{mixed tensor} \end{array}\right\}$$

σ_i = stress vector at P acting across a small surface element whose normal is n_i

σ_{ij} = stress tensor

現在,我們希望由 σ_{ij}, n_i 求出 σ_i

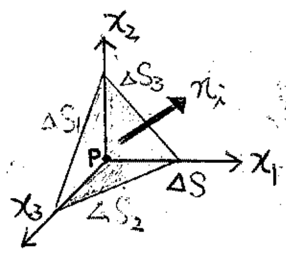


Fig. small tetrahedron 四面體 at point P

f_i = body force per unit mass at P

h = altitude of the tetrahedron measured from the sloping face which is normal to the unit vector n_i

ρ = density at P

$\Delta V = \frac{1}{3} \cdot h \cdot \Delta S$, $\Delta S_1 = \Delta S \cdot n_1 = \Delta S \cdot \cos(n_i, \hat{i})$

$\hat{i} = (1,0,0)$, $n_i = (n_1, n_2, n_3)$., $\hat{i} \cdot n_i = (1,0,0) \cdot (n_1, n_2, n_3) = n_1 = |\hat{i}||n_i| \cos(n_i,$

similarly , $\Delta S_2 = \Delta S \cdot n_2$ & $\Delta S_3 = \Delta S \cdot n_3$

$$\frac{1}{3} \cdot h \cdot \Delta S (\rho + \hat{\rho}) a_i = \Big\{ \frac{1}{3} h \Delta S (\rho + \hat{\rho}) \tilde{f}_i + (\sigma_i + \hat{\sigma}_i) \Delta S$$
$$- (\sigma_{1i} + \hat{\sigma}_{1i}) n_1 \Delta S - (\sigma_{2i} + \hat{\sigma}_{2i}) n_2 \Delta S - (\sigma_{3i} + \hat{\sigma}_{3i}) n_3 \Delta S \Big\}$$

as. $h \to 0$, thus $\hat{\rho} \to 0$, $\hat{\sigma}_i \to 0$, $\hat{\sigma}_{1i} \to 0$, $\hat{\sigma}_{2i} \to 0$, $\hat{\sigma}_{3i} \to 0$

$\Longrightarrow 0 = 0 + \sigma_i - \sigma_{1i} n_1 - \sigma_{2i} n_2 - \sigma_{3i} n_3 \Longrightarrow \sigma_i = \sigma_{ji} n_j$

where σ_{ij} is the second-order tensor & $\sigma_{ji} = \sigma_{ij}$, we get the

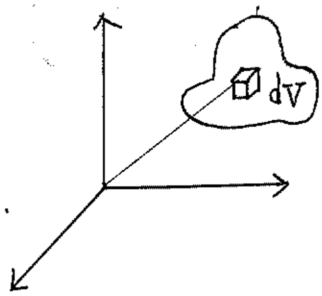
Cauchy's formula $\boxed{\sigma_i = \sigma_{ij} n_j}$

<附> $\iiint_R \vec{\nabla} \cdot \vec{A} \, dV = \oiint_S \vec{A} \cdot \vec{n} \, dS$, $\vec{A}(x,y,z)$: vector field

$$\lim_{N \to \infty} \sum_{i=1}^{N} (\vec{\nabla} \cdot \vec{A})_i \cdot \Delta V_i = \iiint_R (\vec{\nabla} \cdot \vec{A}) \, dV$$

$$\lim_{N \to \infty} \sum_{i=1}^{N} (\vec{A})_i \cdot \vec{n}_i \Delta S_i = \oiint_S (\vec{A} \cdot \vec{n}) \, dS \quad ※$$

moment $\sum \vec{L} = 0$



$$\iiint_R \epsilon_{ijk} x_j f_k \, dV + \oiint_S \epsilon_{ijk} x_j \sigma_k \, dS = 0$$

$\sigma_i = \sigma_{ji} n_j$ & $\sigma_k = \sigma_{sk} n_s$

By divergent theorem :

$$\oiint_S \epsilon_{ijk} x_j \sigma_{sk} n_s \, ds = \iiint_R \frac{\partial(\epsilon_{ijk} x_j \sigma_{sk})}{\partial x_s} dV$$

because $\epsilon_{ijk} \frac{\partial x_j}{\partial x_s} = 0$

$$\iiint_R \frac{\partial(\epsilon_{ijk} x_j \sigma_{sk})}{\partial x_s} dV = \iiint_R \epsilon_{ijk} \left(\sigma_{jk} + x_j \frac{\partial \sigma_{sk}}{\partial x_s} \right) dV$$

$\left(\epsilon_{ijk} \sigma_{jk} = \epsilon_{ijk} \sigma_{sk} \frac{\partial x_j}{\partial x_s} = \epsilon_{isk} \sigma_{sk} \delta_{js} = \epsilon_{isk} \sigma_{jk} \right)$ 加以整理而得

$$\iiint_R \epsilon_{ijk} x_j f_k dV + \oiint_S \epsilon_{ijk} x_j \sigma_{sk} n_s \, ds = 0$$

$$\Rightarrow \iiint_R \epsilon_{ijk} x_j f_k dV + \iiint_R \epsilon_{ijk} \left(\sigma_{jk} + x_j \frac{\partial \sigma_{sk}}{\partial x_s} \right) dV = 0$$

$$\Rightarrow \iiint_R \left[\epsilon_{ijk} x_j \left(f_k + \frac{\partial \sigma_{sk}}{\partial x_s} \right) + \epsilon_{ijk} \sigma_{jk} \right] dV = 0$$

$\Rightarrow \iiint_R \epsilon_{ijk} \sigma_{jk} dV = 0 \Rightarrow \epsilon_{ijk} \sigma_{jk} = 0$, R = any region

$\epsilon_{ijk} \sigma_{jk} = 0$ 代入 $\iiint_R \left[\epsilon_{ijk} x_j \left(f_k + \frac{\partial \sigma_{sk}}{\partial x_s} \right) + \epsilon_{ijk} \sigma_{jk} \right] dV = 0$ 之中

可得 $\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0$

proof : $\underset{0}{\epsilon_{111}} \sigma_{11} + \underset{0}{\epsilon_{112}} \sigma_{12} + \underset{0}{\epsilon_{113}} \sigma_{13} + \underset{0}{\epsilon_{121}} \sigma_{21} + \underset{0}{\epsilon_{122}} \sigma_{21} + \underset{1}{\epsilon_{123}} \sigma_{23}$
$+ \underset{0}{\epsilon_{131}} \sigma_{31} + \underset{-1}{\epsilon_{132}} \sigma_{32} + \underset{0}{\epsilon_{133}} \sigma_{33} = 0$

$\Rightarrow \sigma_{23} - \sigma_{32} = 0 \Rightarrow \sigma_{23} = \sigma_{32}$ ，同理 $\sigma_{13} = \sigma_{31}$ ， $\sigma_{12} = \sigma_{21}$

因此而知 $\sigma_{ij} = \sigma_{ji}$: second-order symmetrical tensor

§2-6 Principle stress and principle axis

n_i : unit vector in the principle axis direction

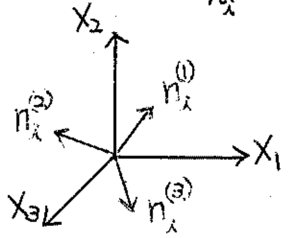
$\sigma_i = \sigma_{ij} n_j$ ， $\sigma_i = \sigma n_i$ （i.e. $\sigma_i \parallel n_i$），$\sigma_{ij} n_j = \sigma n_i = \sigma \delta_{ij} n_j$

$$即 \left\{ \begin{array}{l} \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = \sigma n_1 \\ \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 = \sigma n_2 \\ \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 = \sigma n_3 \end{array} \right\} \Longleftrightarrow \sigma_{ij} n_j = \sigma n_i$$

上述所云得 $(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$ 即 $|\sigma_{ij} - \sigma \delta_{ij}| = 0$

σ : σ_1 ， σ_2 ， σ_3 : principle stress

$n_i^{(1)}$ $n_i^{(2)}$ $n_i^{(3)}$: principle axis



$X_i \longrightarrow X_i'$

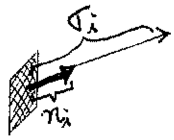
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \longrightarrow \sigma_{ij}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$n_i^{(1)} = (n_{11}, n_{12}, n_{13})$ ， $n_i^{(2)} = (n_{21}, n_{22}, n_{23})$ ， $n_i^{(3)} = (n_{31}, n_{32}, n_{33})$

詳細解說如下所示：

〈i〉 $\sigma_i = \sigma n_i$

$\Rightarrow n_i$: principle axis of σ_{ij}



$\sigma_{ij} n_j - \delta_{ij} \sigma n_j = 0$ i.e. $(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$

$$\Rightarrow \begin{cases} (\sigma_{11}-\sigma)n_1+\sigma_{12}n_2+\sigma_{13}n_3=0 \\ \sigma_{21}n_1+(\sigma_{22}-\sigma)n_2+\sigma_{23}n_3=0 \\ \sigma_{31}n_1+\sigma_{32}n_2+(\sigma_{33}-\sigma)n_3=0 \end{cases}$$

就必須 $\left|\sigma_{ij}-\sigma\delta_{ij}\right|=0$ 即 $\begin{vmatrix} (\sigma_{11}-\sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22}-\sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33}-\sigma) \end{vmatrix}=0$

(解 eigen value) $\Rightarrow \sigma=\sigma_1, \sigma_2, \sigma_3$ ------ principle stress

<ii> substitute $\sigma=\sigma_k$ into $(\sigma_{ij}-\sigma\delta_{ij})n_j=0$

noting that $n_i=n_i^k$: n_i^1, n_i^2, n_i^3 : principle axis & $n_i n_i=1$

<iii> σ_{ij} : real $\Rightarrow \sigma_1, \sigma_2, \sigma_3$: real

$$\sigma_{ij}n_j^k=\sigma_k n_i^k \longrightarrow (1), \quad \overline{\sigma_{ij}}\,\overline{n_j^k}=\overline{\sigma_k}\,\overline{n_i^k} \longrightarrow (2)$$

將 $(1)\times\overline{n_i^k}-(2)\times n_i^k \Rightarrow 0=(\sigma_k-\overline{\sigma_k})\,n_i^k\,\overline{n_i^k} \Rightarrow \sigma_k=\overline{\sigma_k}$

由此可知 σ_k is real

<iv> $\sigma_1, \sigma_2, \sigma_3$ are all distinct (不同的), n_i^1, n_i^2, n_i^3 亦然

i.e. $n_i^s n_i^t=\delta_{st}$ are orthogonal

[說明] $(\sigma_{ij}-\sigma_t\delta_{ij})\,n_j^t=0 \longrightarrow$ ①

$(\sigma_{ij}-\sigma_s\delta_{ij})\,n_j^s=0 \longrightarrow$ ②

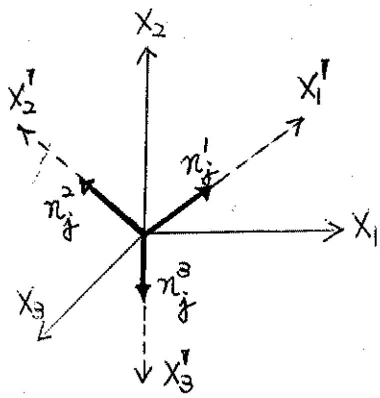
將 ①$\times n_i^s$ − ②$\times n_i^t \Rightarrow (-\sigma_t+\sigma_s)\delta_{ij}n_i^t n_j^s=0 \Rightarrow (\sigma_s-\sigma_t)n_i^t n_i^s=0$

if $s\neq t$ then $\sigma_s\neq\sigma_t$ in order that $n_i^t n_i^s=0$

implies n_i^1, n_i^2, n_i^3 are orthogonal

gain $\underline{(\sigma_{ij}-\delta_{ij}\sigma_t)\,n_j^t n_i^s=0}$

〈V〉 choose x_1', x_2', x_3' axis to coinside with the principle axis n_j^1, n_j^2, n_j^3



transformation matrix $C_{ij}=\begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \\ n_1^3 & n_2^3 & n_3^3 \end{bmatrix}=n_j^i$

[註] C_{ij} 是 $\cos\theta_{ij}$ 即 i' 軸和 j 軸的夾角 ※

$$x_i \longrightarrow \underset{\text{principle axis}}{x_i'} \qquad \sigma_{ij} \longrightarrow \sigma_{ij}'$$

$$X_i' = C_{ij} X_j \Longrightarrow X_i' = n_j^i X_j$$

$\sigma_{ij}' = C_{is} C_{jt} \sigma_{st} = n_s^i n_t^j \sigma_{st}$, $\sigma_{ij} n_j^k = \sigma_k \delta_{ij} n_j^k = \sigma_k n_i^k$ (k: no summation)

$\sigma_{ij}' = C_{is} C_{jt} \sigma_{st} = n_s^i n_t^j \sigma_{st} = n_s^i \sigma_j n_s^j$ (j: no summation) $= n_s^i n_s^j \sigma_j = \delta_{ij}\sigma_j$

$\sigma_{ij}' = \delta_{ij}\sigma_j = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$,, [註] $\underset{\text{principle vector}}{\sigma_i} = \underset{\text{principle stress}}{\sigma}\, n_i \rightarrow$ normal vector (principle axis) ※

[problem] Given $\sigma_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(a) find principle stress

(b) direction cosines of the principle axis (C_{ij})

solution: (a) $\left|(\sigma_{ij} - \delta_{ij}\sigma_t)\right| = \left|\begin{pmatrix} 0-\sigma & 0 & 0 \\ 0 & 0-\sigma & 1 \\ 0 & 1 & 0-\sigma \end{pmatrix}\right| = 0$

$\Rightarrow -\sigma^3+\sigma=0 \Rightarrow \sigma(1-\sigma^2)=0 \Rightarrow \sigma = 0, 1, -1$: eigen value

即 principle stress 分別為 $0, +1, -1$ (即 $\sigma_1=0, \sigma_2=1, \sigma_3=-1$)

(b) 求出 eigen vector

[註] t is any real number ※

$\boxed{1}$ $\begin{pmatrix} 0-0 & 0 & 0 \\ 0 & 0-0 & 1 \\ 0 & 1 & 0-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$ 選 unit vector $n_1^i=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

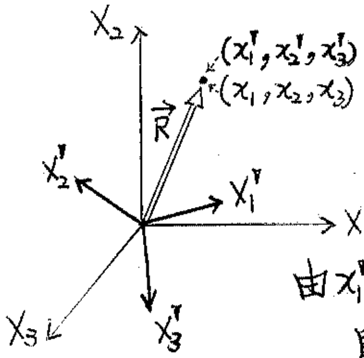
$\boxed{2}$ $\begin{pmatrix} 0-1 & 0 & 0 \\ 0 & 0-1 & 1 \\ 0 & 1 & 0-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ t \\ t \end{pmatrix}$ 選 unit vector $n_2^i=\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\boxed{3}$ $\begin{pmatrix} 0-(-1) & 0 & 0 \\ 0 & 0-(-1) & 1 \\ 0 & 1 & 0-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ t \\ -t \end{pmatrix}$ 選 unit vector $n_3^i=\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

由 $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ & $C_{ij}=n_j^i=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

principle stress 分别是 $\left\{\begin{aligned} n_j^1 &= (1, 0, 0) \\ n_j^2 &= (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) \\ n_j^3 &= (0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}) \end{aligned}\right\}$ ※

§2-7 Transformation law for stress tensor



$$\vec{R}\begin{cases} = x_1\hat{i}+x_2\hat{j}+x_3\hat{k} \ : \ x_i \\ = x_1'\hat{i}'+x_2'\hat{j}'+x_3'\hat{k}' \ : \ x_i' \end{cases}$$

θ_{ij} : angles between x_i' and x_j axis

而 $C_{ij} = \cos\theta_{ij}$

由 $x_1' = x_1\cos\theta_{11}+x_2\cos\theta_{12}+x_3\cos\theta_{13} = C_{11}x_1+C_{12}x_2+C_{13}x_3 = C_{1j}x_j$

歸納而得 $\boxed{x_i' = C_{ij}x_j}$

(或由 $x_1\hat{i}+x_2\hat{j}+x_3\hat{k} = x_1'\hat{i}'+x_2'\hat{j}'+x_3'\hat{k}'$ 兩邊 dot $\hat{i}'$

得 $x_1\hat{i}'\cdot\hat{i}+x_2\hat{i}'\cdot\hat{j}+x_3\hat{i}'\cdot\hat{k} = x_1'\hat{i}'\cdot\hat{i}'+x_2'\hat{i}'\cdot\hat{j}'+x_3'\hat{i}'\cdot\hat{k}'$

知 $x_1\cos\theta_{11}+x_2\cos\theta_{12}+x_3\cos\theta_{13} = x_1'+0+0 = x_1'$)

反變換時，$x_1\hat{i}+x_2\hat{j}+x_3\hat{k} = x_1'\hat{i}'+x_2'\hat{j}'+x_3'\hat{k}'$ 兩邊 dot $\hat{i}$

$$\Longrightarrow x_1 = x_1'C_{11}+x_2'C_{21}+x_3'C_{31} \Rightarrow \boxed{x_i = C_{ji}x_j'}$$

或 $X' = \mathbb{C}X$, $\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$: transformation matrix

[Ex] 我們知道 $C_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 即為

在 xy 平面旋轉 θ 角的變換矩陣 ※

transformation law for coordinate $\boxed{\begin{aligned} x_i' &= C_{ij}x_j \\ x_j &= C_{kj}x_k' \end{aligned}}$

$$\Rightarrow \begin{cases} x_i' = C_{ij}C_{kj}x_k' \\ \quad \parallel \\ \delta_{ik}x_k' = C_{ij}C_{kj}x_k' \end{cases} \Rightarrow (C_{ij}C_{kj}-\delta_{ik})x_k' \equiv 0$$

$$\Rightarrow \boxed{C_{ij}C_{kj} = \delta_{ik}} \ : \text{orthogonal transformation}$$

[說明] $\left\{\begin{array}{l} x_1' = C_{11}x_1 + C_{12}x_2 + C_{13}x_3 \\ x_2' = C_{21}x_1 + C_{22}x_2 + C_{23}x_3 \\ x_3' = C_{31}x_1 + C_{32}x_2 + C_{33}x_3 \end{array}\right\}$, $\left\{\begin{array}{l} x_1 = C_{11}x_1' + C_{21}x_2' + C_{31}x_3' \\ x_2 = C_{12}x_1' + C_{22}x_2' + C_{32}x_3' \\ x_3 = C_{13}x_1' + C_{23}x_2' + C_{33}x_3' \end{array}\right\}$

$x_1' = C_{11}C_{11}x_1' + C_{11}C_{21}x_2' + C_{11}C_{31}x_3' + C_{12}C_{12}x_1' + C_{12}C_{22}x_2' + C_{12}C_{32}x_3' + C_{13}C_{13}x_1' + C_{13}C_{23}x_2' + C_{13}C_{33}x_3'$ ※

Similarly , $C_{ji}C_{jk} = \delta_{ik}$

令 C_{ij} 的行列式值為 d，即 $|C_{ij}| = d$，則 $|C_{ji}| = d$

$\Rightarrow |C_{ij}||C_{ji}| = d^2 \Rightarrow |C_{ik}C_{jk}| = d^2 \Rightarrow |\delta_{ij}| = d^2 = 1$

$\Rightarrow |C_{ij}| \left\{\begin{array}{l} = +1 \dashrightarrow \text{proper orthogonal transformation} \\ = -1 \dashrightarrow \text{improper orthogonal transformation} \end{array}\right\}$

(b) definition of tensor

<i> zeroth-order tensor (scalar)

number of component : $3^0 = 1$

$x_i \rightarrow x_i'$ $\quad \phi \rightarrow \phi'$

$x_i' = C_{ij}x_j$ $\quad \phi' = \phi$

<ii> first-order tensor (vector)

number of component : $3^1 = 3 \dashrightarrow a_i$

$x_i \rightarrow x_i'$ $\quad a_i \rightarrow a_i'$

$x_i' = C_{ij}x_j$ $\quad a_i' = C_{ij}a_j$

<iii> second-order tensor (dyadic)

number of component : $3^2 = 9 \dashrightarrow a_{ij}$

$x_i \rightarrow x_i'$ $\quad a_{ij} \rightarrow a_{ij}'$

$x_i' = C_{ij}x_j$ $\quad a_{ij}' = C_{is}C_{jt}a_{st}$

<iv> nth-order tensor

number of component : 3^n - - - - > $a_{i_1 i_2 \cdots i_n}$

$x_i \longrightarrow x_i'$ $\qquad a_{i_1 i_2 \cdots i_n} \longrightarrow a'_{i_1 i_2 \cdots i_n}$

$x_i' = c_{ij} x_j$ $\qquad a'_{i_1 i_2 i_3 \cdots i_n} = c_{i_1 j_1} c_{i_2 j_2} c_{i_3 j_3} \cdots c_{i_n j_n} a_{j_1 j_2 j_3 \cdots j_n}$

(c) Cauchy formula

$\sigma_i = \sigma_{ij} n_j$ & $\sigma_i' = \sigma_{ij}' n_j' \Rightarrow \sigma_i' = c_{ij} \sigma_j = \sigma_{ij}' n_j' \Rightarrow c_{ij} \sigma_{jk} n_k = \sigma_{ij}' n_j'$

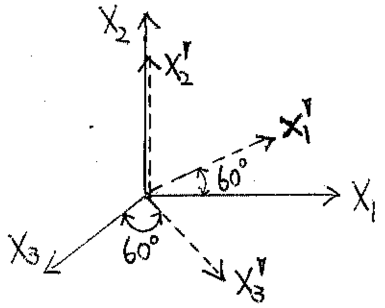
[test] 〈i〉求 x_i 與 x_i' 座標間之 transformation matrix C_{ij}

x_i' 為由 x_i 對 x_2 軸旋轉 $60°$ 而得

〈ii〉若 $\sigma_{ij} = \begin{pmatrix} 1 & -3 & -1 \\ -3 & 2 & 2 \\ -1 & 2 & -1 \end{pmatrix}$ dynes/cm² 求 σ_{ij}'

solution：

〈i〉 $C_{ij} = \cos\theta_{ij} = \begin{pmatrix} \cos 60° & \cos 90° & \cos 210° \\ \cos 90° & \cos 0° & \cos 90° \\ \cos 30° & \cos 90° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$



〈ii〉

$\sigma_{ij}' = C_{is} C_{jt} \sigma_{st} = \begin{pmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{pmatrix}$

如 σ_{13}' 的求法：$\sigma_{13}' = C_{1s} C_{3t} \sigma_{st}$

即 $\sigma_{13}' = C_{11}C_{31}\sigma_{11} + C_{11}C_{32}\sigma_{12} + C_{11}C_{33}\sigma_{13}$

$+ C_{12}C_{31}\sigma_{21} + C_{12}C_{32}\sigma_{22} + C_{12}C_{33}\sigma_{23}$

$+ C_{13}C_{31}\sigma_{31} + C_{13}C_{32}\sigma_{32} + C_{13}C_{33}\sigma_{33}$ ※

§ 2-8 The condition of incompressibility

conservation of mass : $m = V_0 \rho_0$

$m = (V_0 + \Delta V)(\rho_0 + \Delta\rho) = V_0\rho_0 + V_0\Delta\rho + \rho_0\Delta V + \Delta\rho\Delta V \Rightarrow \frac{\Delta\rho}{\rho_0} = -\frac{\Delta V}{V_0}$

E : modulus of elasticity (係數), $\Delta P = -E\frac{\Delta V}{V_0}$, E愈大愈不易壓縮

$\rho = \rho(P, \underset{\uparrow \text{ entropy}}{S})$ $\quad \Delta\rho = \left(\frac{\partial\rho}{\partial P}\right)_S \Delta P + \left(\frac{\partial\rho}{\partial S}\right)_P \Delta S$

for adiabatic process $\Delta S = 0$, $\Delta\rho = \left(\frac{\partial\rho}{\partial P}\right)_S \Delta P$

by Bernoulli equation $P + \frac{1}{2}\rho v^2 = \text{constant}$

$(\Delta P)_{max} \sim \frac{1}{2}\rho v^2 \sim \rho v^2$, $\Delta\rho = \left(\frac{\partial\rho}{\partial P}\right)_S \rho v^2$, v : characteristic velocity

又 $\left(\frac{\partial\rho}{\partial P}\right)_S = \frac{1}{c^2}$, c: speed of sound ∴ $\Delta\rho = \frac{\rho v^2}{c^2}$ for incompressibility

故 $\frac{\Delta\rho}{\rho} = \frac{v^2}{c^2}$ ∴ $\frac{\Delta\rho}{\rho} \ll 1$, $\frac{v^2}{c^2} \ll 1$, $\frac{v}{c} \ll 1$

$M = \frac{v}{c}$: dimensional parameter : Mach number (馬赫)

§ 2-9 The stress tensor in a fluid at rest

If fluid is rest, then $\sigma_{ij} = -P\delta_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$

P : static pressure

$|\sigma_{ij} - \sigma\delta_{ij}| = 0 \Rightarrow (\sigma + P)^3 = 0 \Rightarrow \sigma = -P, -P, -P$

☆☆☆ 期中考模擬復習 ☆☆☆

1. 試解釋 σ_{st}

sol：$\sigma_{st} = \sigma_t(\hat{s})$ 即作用面垂直 $\hat{s}$ 且作用於 $\hat{t}$ 方向的力。

2. 說明 $\vec{L}$，$\vec{H}$，$\dot{P}_i$，$\dot{H}_i$

sol：$\vec{L} = \vec{r} \times \vec{F}$，$\vec{H} = \vec{r} \times \vec{p}$，$\dot{P}_i = \dfrac{dP_i}{dt} = F_i$，$\dot{H}_i = \dfrac{dH_i}{dt} = L_i$

3. 試寫出流体中，F的分解及L的分解

sol：

$$F = \frac{d}{dt}\left[\iiint_R \rho v_i \, dV\right] = \iiint_R f_i \, dV + \iint_S \sigma_i \, ds$$

$$L = \frac{d}{dt}\iiint_R \epsilon_{ijk} x_j v_k \rho \, dV = \iiint_R \epsilon_{ijk} x_j f_k \, dV + \iint_S \epsilon_{ijk} x_j \sigma_k \, ds$$

4. 試說明 δ_{ij}，ϵ_{ijk} 的定義，並算出 δ_{ii} 及 $\epsilon_{ijk}\epsilon_{ijk}$

sol：$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$，$\epsilon_{ijk}\epsilon_{ijk} = \begin{cases} 1, & 1,2,3\text{的偶排列} \\ -1, & 1,2,3\text{的奇排列} \\ 0, & \text{其他} \end{cases}$，$\begin{cases} \delta_{ii} = 3 \\ \epsilon_{ijk}\epsilon_{ijk} = 6 \end{cases}$

5. 試以 index notation 寫出 $\vec{\nabla}\phi$，$\vec{\nabla}\cdot\vec{A}$，$\vec{\nabla}\times\vec{A}$，$\vec{A}\times\vec{B}$，$\vec{A}\cdot\vec{B}$，$[A][X]=[B]$

sol：$\vec{\nabla}\phi = \dfrac{\partial\phi}{\partial x_i}$，$\vec{\nabla}\cdot\vec{A} = \dfrac{\partial a_i}{\partial x_i}$，$\vec{\nabla}\times\vec{A} = \epsilon_{ijk}\dfrac{\partial a_k}{\partial x_j}$，$\vec{A}\cdot\vec{B} = a_i b_i$

$\vec{A}\times\vec{B} = \epsilon_{ijk} a_j b_k$，$[A][X]=[B] \iff a_{ij} x_j = b_i$

6. 證明 Cauchy's formula $\sigma_i = \sigma_{ji} n_j$

sol：由 $\Delta V = \frac{1}{3} h \Delta S$，$\Delta S_1 = n_1 \Delta S$，$\Delta S_2 = n_2 \Delta S$，$\Delta S_3 = n_3 \Delta S$

還有 $\frac{1}{3} h \Delta S (\rho + \hat{\rho}) a_i - \frac{1}{3} h \Delta S (\rho + \hat{\rho}) \hat{f}_i$

$$= (\sigma_i + \hat{\sigma}_i)\Delta S - (\sigma_{1i} + \hat{\sigma}_{1i}) n_1 \Delta S - (\sigma_{2i} + \hat{\sigma}_{2i}) n_2 \Delta S - (\sigma_{3i} + \hat{\sigma}_{3i}) n_3 \Delta S$$

並令 $\hat{\rho} \to 0$，$\hat{\sigma}_i \to 0$，$\hat{\sigma}_{1i} \to 0$，$\hat{\sigma}_{2i} \to 0$，$\hat{\sigma}_{3i} \to 0$（當 $h \to 0$ 之時），即可得證 ※

7. 請由 $\Sigma F_i = 0$ 導出 $\dfrac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0$

sol:

$$\Sigma F_i = 0 \Rightarrow \iiint_R f_i\, dV + \iint_S \sigma_i\, ds = 0 \Rightarrow \iiint_R f_i\, dV + \iint_S \sigma_{ji} n_j\, ds = 0$$

$$\Rightarrow \iiint_R f_i\, dV + \iiint_R \frac{\partial \sigma_{ji}}{\partial x_j}\, dV = 0 \Rightarrow \iiint_R \left[f_i + \frac{\partial \sigma_{ji}}{\partial x_j}\right] dV = 0 \Rightarrow f_i + \frac{\partial \sigma_{ji}}{\partial x_j} = 0$$

8. 請由 $\Sigma L = 0$ 導出 $\dfrac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0$ 及 $\sigma_{ij} = \sigma_{ji}$

sol:

$$\Sigma L = 0 \Rightarrow \iiint_R \epsilon_{ijk} x_j f_k\, dV + \oiint_S \epsilon_{ijk} x_j \sigma_k\, ds = 0$$

又 $\displaystyle\oiint_S \epsilon_{ijk} x_j \sigma_k\, ds = \oiint_S \epsilon_{ijk} x_j \sigma_{sk} n_s\, ds = \iiint_R \frac{\partial(\epsilon_{ijk} x_j \sigma_{sk})}{\partial x_s}\, dV = \iiint_R \epsilon_{ijk}\left(\sigma_{jk} + x_j \frac{\partial \sigma_{sk}}{\partial x_s}\right) dV = 0$

$$\Rightarrow \iiint_R \left[\epsilon_{ijk} x_j \left(f_k + \frac{\partial \sigma_{sk}}{\partial x_s}\right) + \epsilon_{ijk} \sigma_{jk}\right] dV = 0$$ ，其中 $\dfrac{\partial x_j}{\partial x_s} = \sigma_{js}$

$$\left\{\begin{aligned} &\Rightarrow \iiint_R \epsilon_{ijk} \sigma_{jk}\, dV = 0 \Rightarrow \epsilon_{ijk} \sigma_{jk} = 0 \Rightarrow \sigma_{ij} = \sigma_{ji} \\ &\Rightarrow \iiint_R \left(\frac{\partial \sigma_{sk}}{\partial x_s} + f_k\right) dV = 0 \Rightarrow \frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0 \end{aligned}\right\}$$ ※

9. 試導出 $|\sigma_{ij} - \sigma \delta_{ij}| = 0$

sol: $\sigma \delta_{ij} n_j = \sigma n_i = \sigma_i = \sigma_{ij} n_j \Rightarrow (\sigma_{ij} - \sigma \delta_{ij}) n_j = 0 \Rightarrow |\sigma_{ij} - \sigma \delta_{ij}| = 0$ ※

10. 寫出 principle stress σ_j 和 principle axis n_i^j 的對應關係。

sol:

$$\left\{\begin{aligned} \sigma_1 &\longrightarrow n_i^1 = (n_{11}, n_{12}, n_{13}) \\ \sigma_2 &\longrightarrow n_i^2 = (n_{21}, n_{22}, n_{23}) \\ \sigma_3 &\longrightarrow n_i^3 = (n_{31}, n_{32}, n_{33}) \end{aligned}\right\}$$ ※

11、什麼是 C_{ij}？請寫出 transformation law for coordinate 並加以導出 orthogonal transformation．

sol：$C_{ij} = \cos\theta_{ij}$，其中 θ_{ij} 是 x_i' 和 x_j 的夾角

transformation law for coordinate $\left\{ \begin{array}{l} x_i' = C_{ij} x_j \\ x_j = C_{kj} x_k' \end{array} \right\}$

$$\Rightarrow \left\{ \begin{array}{l} x_i' = C_{ij} C_{kj} x_k' \\ \delta_{ik} x_k' = C_{ij} C_{kj} x_k' \end{array} \right\} \Rightarrow (C_{ij} C_{kj} - \delta_{ik}) x_k' = 0$$

$\Rightarrow$ orthogonal transformation $C_{ij} C_{kj} = \delta_{ik}$ ※

12、導出 Cauchy formula $\sigma_{ij}' = C_{is} C_{jt} \sigma_{st}$

sol：$\sigma_i = \sigma_{ij} n_j$，$\sigma_i' = \sigma_{ij}' n_j'$，$\sigma_i' = C_{ij} \sigma_j \Rightarrow C_{ij} \sigma_j = \sigma_{ij}' n_j'$

$$\Rightarrow C_{ij} \sigma_{jk} n_k = \sigma_{ij}' n_j' \Rightarrow C_{ij} \sigma_{jk} C_{sk} n_s' = \sigma_{ij}' n_j' = \sigma_{is}' n_s'$$

$$\Rightarrow (\sigma_{is}' - C_{ij} C_{sk} \sigma_{jk}) n_s' = 0 \Rightarrow \sigma_{is}' = C_{ij} C_{sk} \sigma_{jk}$$

$$\Rightarrow \sigma_{ij}' = C_{is} C_{jt} \sigma_{st}$$ ※

13、完成下列恒等式：

(1) $\vec{\nabla} \cdot (\vec{F} \times \vec{G}) = ?$ (2) $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = ?$ (3) $\vec{\nabla} \times (\vec{\nabla} \phi) = ?$ (4) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = ?$

sol：(1) $\vec{\nabla} \cdot$

(2) $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{F}) - \vec{\nabla}^2 \vec{F}$

(3) $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$ (4) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ ※

14、寫出 Divergence theorem 及 Stokes theorem

sol：Divergence theorem $\iiint_R \vec{\nabla} \cdot \vec{A} \, dV = \iint_S \vec{n} \cdot \vec{A} \, dS$

Stokes theorem $\iint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{n} \, d\sigma = \int_C \vec{A} \cdot \vec{t} \, dl$ ※

15、請導出 equation of continuity.

sol: mass out the region $\Delta x \Delta y \Delta z$ per unit time

$$\left[\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right]\Delta x \Delta y \Delta z=\left[\vec{\nabla}\cdot(\rho\vec{v})\right]\Delta x \Delta y \Delta z$$

$\vec{v}$: velocity of fluid，又 $\frac{\partial m}{\partial t}=\frac{\partial(\rho \Delta x \Delta y \Delta z)}{\partial t}$

∵ conservation of mass $\left[\vec{\nabla}\cdot(\rho\vec{v})\right]\Delta x \Delta y \Delta z=-\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$

$\therefore \frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0$ ※

16、請導出 $\vec{\omega}=\frac{1}{2}\vec{\nabla}\times\vec{F}$

sol: $\vec{F}=F_x\hat{i}+F_y\hat{j}+F_z\hat{k}$，$\vec{U}_\theta=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}+0\,\hat{k}$

$$\overline{\vec{F}\cdot\vec{U}_\theta}=\frac{1}{2\pi}\int_0^{2\pi}\vec{F}\cdot\vec{U}_\theta\, d\theta=\frac{1}{2}r\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

$$\omega_z=\frac{\overline{\vec{F}\cdot\vec{U}_\theta}}{r}=\frac{1}{2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\text{，similarly}$$

$$\omega_y=\frac{1}{2}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\text{，}\omega_x=\frac{1}{2}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)$$

故 $\vec{\omega}=\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}=\frac{1}{2}\vec{\nabla}\times\vec{F}$ ※

17、試由 $C_{ij}=\begin{bmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$ 和 $\sigma_{ij}=\begin{bmatrix}1&0&0\\-1&0&1\\0&1&-1\end{bmatrix}$ 求出 $|\sigma'_{ij}|$

sol: $|\sigma'_{ij}|=|C_{is}C_{jt}\sigma_{st}|=|C_{is}||C_{jt}||\sigma_{st}|=1\times1\times(-1)=-1$ ※

18、請用一種方法證記 $\sigma_{ij}n_j^k=\sigma_k n_i^k$

sol: 由 $\sigma_i=\sigma_{ij}n_j$ 和 $\sigma_i=\sigma n_i \Rightarrow \sigma_{ij}n_j=\sigma n_i$

當 $\sigma=\sigma_k$ 則 $n_j=n_j^k$ 及 $n_i=n_i^k \Rightarrow \sigma_{ij}n_j^k=\sigma_k n_i^k$ ※

19. 試述 The stress tensor in a fluid at rest.

sol: if fluid is rest $\Rightarrow \sigma_{ij} = -P\delta_{ij}$

P: static pressure, $\sigma_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$ ※

20. 試導出 $\frac{\Delta \rho}{\rho_0} = -\frac{\Delta V}{V_0}$

sol: from the conservation of mass $m = V_0 \rho_0$

and $m = (V_0 + \Delta V)(\rho_0 + \Delta \rho)$ we get $\frac{\Delta \rho}{\rho_0} = -\frac{\Delta V}{V_0}$ ※

21. $\sigma_i = \sigma n_i$, σ_i, σ, n_i 如何稱呼?

sol: $\left\{ \begin{array}{l} \sigma_i \dashrightarrow \text{principle vector} \\ \sigma \dashrightarrow \text{principle stress} \\ n_i \dashrightarrow \text{principle axis} \end{array} \right\}$

22. 試述 C_{ij} 與 n_j^i 的關係

sol: transformation matrix $C_{ij} = \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \\ n_1^3 & n_2^3 & n_3^3 \end{bmatrix} = n_j^i$ ※

23. 要如何從 σ_{ij} 求出 σ_{ij}'?

sol: 從 $|\sigma_{ij} - \sigma\delta_{ij}| = 0$ 求出 $\sigma = \sigma_1, \sigma_2, \sigma_3$ (即求 eigen value 之法)

即 $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \longrightarrow \sigma_{ij}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$ ※

24. 什麼是 flux? 什麼是 divergence?

sol: $\frac{\text{volumn cross area } \delta S}{\text{unit time}} = \vec{F} \cdot \vec{n}\, \delta S = \frac{\delta V}{\delta t}$

$\dashrightarrow$ flux of the vector field $\vec{F}$ through the area δS

而 divergence of $\vec{F} = \frac{\text{total flux}}{\text{unit volumn}} = \vec{\nabla} \cdot \vec{F}$ ※