正壓不穩定之必要條件探討

徐天佑 劉廣英 曾鴻陽 張怡蕙
中國文化大學

Abstract

There are many different kinds of instability types. In subtropics, the development of a synoptic system is associated with baroclinic instability. In tropic, the development of a synoptic system is associated with barotropic instability.

The theory of baroclinic instability is well developed which is related to the temperature gradient. The barotropic instability is influenced by the horizontal wind shear. And its necessary condition is that the absolute vorticity of mean wind flow at somewhere ($Us = U(y)$) equal to zero. If anywhere there is satisfying the necessary condition weather the disturbance is developed or undeveloped. It must be also depend on the basic flow not all the condition absolute vorticity $(\beta - d^2U/dy^2) = 0$ could become barotropic instability. Except the absolute vorticity equal zero, and the basic flow must satisfy $(d^2U/dy^2) \times (U - C) \leq 0$ and the wave speed relative to the mean flow 。

Introduction

In barotroptic atmospheric system, weather system changes are related to the vorticity variation.

The vorticity equation of barotroptic system as show fellow

$$(\overline{U} - C)(d^2\Psi/dy^2 - k^2\Psi) + (\beta - (d^2\overline{U}/dy^2)\Psi = 0$$

Ψ： stream function, $\overline{U}$ ：mean flow

$\beta \equiv df/dy$, variation of the Coriolis parameter with latitude

C：wave speed, k：wave number

A necessary condition of barotropic instability is that at some value of y_k ,

$$(\beta - (d^2\overline{U}/dy^2) = 0 \qquad -d < y_k < d$$

This theorem was derived by Kuo(1951). But not all the necessary condition of absolute vorticity that equal to zero and the instability will development. Tollmien(1935) has proven in a non-viscous flow if there is an inflection point the flow also instable. What other condition will cause the disturbance develop or undeveople traveling in a basic flaw that is the purpose of our research?

Theoretical discussion

From barotropical vorticity equation

$$(\overline{U} - C)(d^2\Psi/dy^2 - k^2\Psi) + (\beta - (d^2\overline{U}/dy^2)\Psi = 0$$

We have the flowing discussion：

1. If $(\overline{U} - C)$ equal zero, from the above equation.

   Then $(\overline{U} - C)(d^2\Psi/dy^2 - k^2\Psi) + (\beta - (d^2\overline{U}/dy^2)\Psi = 0$

   There is non-trivial solution.

2. If $(\overline{U} - C)$ is not equal to zero,

   then $d^2\Psi/dy^2 +$ 【 $(\beta - (d^2\overline{U}/dy^2)/(\overline{U} - C)) - k^2$ 】 $\Psi = 0$

   Let m= 【 $(\beta - (d^2\overline{U}/dy^2)/(\overline{U} - C)) - k^2$ 】

(1) $m > 0$ the wave is stable

(2) $m < 0$ the amplitude of a wave is unstable which is intensify or weakness.

That is 【$(\beta - d^2\overline{U}/dy^2)/(\overline{U} - C)) - k^2$ 】 < 0

$$(\beta - d^2\overline{U}/dy^2)/(\overline{U} - C)) < k^2$$

For simple, let the flow be nonrotate flow, then $-(d^2\overline{U}/dy^2)/(\overline{U}-C)<k^2$

$\because k^2>0$

$\therefore -(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$

Then $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

A. If $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

(A) $\overline{U}(y)$ monotony decrease and $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

a. If $(\overline{U}(y)-C)>0$ that is the wave travels less than mean flow

if $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$ then $(d^2\overline{U}/dy^2)(\overline{U}-C)/(\overline{U}-C)^2>0$

that is $(d^2\overline{U}/dy^2)(\overline{U}-C)>0$

$d^2\overline{U}/dy^2$ and $\overline{U}-C$ are in same sign.

IF $d^2\overline{U}/dy^2>0$ and $\overline{U}(y)-C>0$

As $d^2\overline{U}/dy^2>0$ that means $-(-d^2\overline{U}/dy^2)>0$, then $-(d\zeta/dy)>0$ and that is $d\zeta/dy<0$, the gradient of vorticity is decrease along y axi, and U(y) is monotony decrease along y and $\overline{U}(y)-C>0$. That means flow has positive vorticity.

That is the wind field offer positive vorticity to the perturbation along y axi. But the gradient of vorticity offer negative vorticity along y axis. So the environment is not benefit to the perturbation. And the perturbation is stable.

b. If $(\overline{U}(y)-C)<0$

$(\overline{U}(y)-C)<0$ and $d^2\overline{U}/dy^2<0$

Then $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

As $d^2\overline{U}/dy^2<0$ that means $-(-d^2\overline{U}/dy^2)<0$, then $-(d\zeta/dy)<0$ and that is $d\zeta/dy>0$, the gradient of vorticity is increase along y axi, but $\overline{U}$ (y) is less then C along y. The phase speed of wave relative to the mean wind field has negative vorticity along y axis. But the gradient of vorticity along y axis is positive. So the environment is not benefit to the perturbation. And the perturbation is stable.

B If $(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$

(A) $\overline{U}$ (y) decrease monotony and $|(d^2\overline{U}/dy^2)/(\overline{U}-C)|<k^2$

a. $(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$ but $\overline{U}(y)-C>0$

then $(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$

that means $d^2\overline{U}/dy^2$ and $\overline{U}-C$ are in opposite sign.

IF $d^2\overline{U}/dy^2<0$ and $\overline{U}(y)-C>0$

As $d^2\overline{U}/dy^2<0$ that means $-(-d^2\overline{U}/dy^2)<0$, then $-(d\zeta/dy)<0$ and that is $d\zeta/dy>0$, the gradient of vorticity is increase along y axi, and U(y) is monotony decrease along y but $\overline{U}(y)-C>0$. That means the phase speed of wave relative to the mean wind field has positive vorticity.

That is the wind field offer positive vorticity to the perturbation along y axi. But the gradient of vorticity also offers positive vorticity along y axis. So the environment is benefit to the perturbation. And the perturbation is unstable.

b. $(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$ but $\overline{U}(y)-C<0$

then $(d^2\overline{U}/dy^2)/(\overline{U}-C)<0$

that means $d^2\overline{U}/dy^2$ and $\overline{U}-C$ are in opposite sign.

IF $d^2\overline{U}/dy^2>0$ and $\overline{U}(y)-C<0$

As $d^2\overline{U}/dy^2>0$ that means $-(-d^2\overline{U}/dy^2)>0$, then $-(d\zeta/dy)>0$ and that is $d\zeta/dy<0$, the gradient of vorticity is decrease along y axi, and U(y) is monotony decrease along y, wind of $\overline{U}$ (y) is in vorticity state, but $\overline{U}(y)-C<0$, That means the phase speed of wave relative to the mean wind field has negative vorticity.

That is the wind field offer negative vorticity to the perturbation along y axi. But the gradient of vorticity is also decrease along y axis. So the environment is benefit to the antivorticity. And the antivorticity is unstable.

C $\overline{U}(y)$ monotony increase and $(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

a. If $(\overline{U}(y)-C)>0$ that is the wave travels less than mean flow

$(d^2\overline{U}/dy^2)/(\overline{U}-C)>0$

and $|(d^2\bar{U}/dy^2)/(\bar{U}-C)| < k^2$

if $(d^2\bar{U}/dy^2)/(\bar{U}-C)>0$ then $(d^2\bar{U}/dy^2)(\bar{U}-C)/(\bar{U}-C)^2 >0$

that is $(d^2\bar{U}/dy^2)(\bar{U}-C)>0$

$d^2\bar{U}/dy^2$ and $\bar{U}-C$ are in same sign.

IF $d^2\bar{U}/dy^2>0$ and $\bar{U}(y)-C>0$

As $d^2\bar{U}/dy^2>0$ that means $-(-d^2\bar{U}/dy^2)>0$, then $-(d\zeta/dy)>0$ and that is $d\zeta/dy<0$, the gradient of vorticity is decreas increase along y axi, and U(y) is monotony increase along y. That means the phase speed of wave relative to the mean wind field has negative vorticity.

That is the wind field offer negative vorticity to the perturbation along y axi. But the gradient of vorticity is decrease along y axis. So the environment is benefit to the antivorticity. And the antivorticity is unstable.

b. If $(\bar{U}(y)-C)<0$

$(\bar{U}(y)-C)<0$ and $d^2\bar{U}/dy^2<0$

Then $(d^2\bar{U}/dy^2)/(\bar{U}-C)>0$

As $d^2\bar{U}/dy^2<0$ that means $-(-d^2\bar{U}/dy^2)<0$, then $-(d\zeta/dy)<0$ and that is $d\zeta/dy>0$, the gradient of vorticity is increase along y axi, but $\bar{U}$ (y) is less then C along y. The phase speed of wave relative to the mean wind field has negative vorticity along y axis. But the gradient of vorticity along y axis is positive. So the environment is not benefit to the perturbation. And the perturbation is stable.

(D) $\bar{U}(y)$ monotony increase

$|(d^2\bar{U}/dy^2)/(\bar{U}-C)| < k^2$

And $(d^2\bar{U}/dy^2)/(\bar{U}-C)<0$

a. $(d^2\bar{U}/dy^2)/(\bar{U}-C)<0$ but $\bar{U}(y)-C<0$

then $(d^2\bar{U}/dy^2)/(\bar{U}-C)<0$

that means $d^2\bar{U}/dy^2$ and $\bar{U}-C$ are in opposite sign.

IF $d^2\bar{U}/dy^2>0$ and $\bar{U}(y)-C<0$

As $d^2\bar{U}/dy^2>0$ that means $-(-d^2\bar{U}/dy^2)>0$, then $-(d\zeta/dy)>0$ and that is $d\zeta/dy<0$, the gradient of vorticity is decrease along y axi, and U(y) is monotony increase along y, so the wind of $\bar{U}(y)$ is in antivorticity state, and the wave to the means wind field is also with antivorticity.

That is disturbance to the wind field along y axi is negatietive. And the gradient of vorticity is decrease along y axis. So the environment is benefit to the antivorticity. And the antivorticity is unstable.

b. $(d^2\bar{U}/dy^2)/(\bar{U}-C)<0$ but $\bar{U}(y)-C>0$

then $(d^2\bar{U}/dy^2)/(\bar{U}-C)<0$

that means $d^2\bar{U}/dy^2$ and $\bar{U}-C$ are in opposite sign.

IF $d^2\bar{U}/dy^2<0$ and $\bar{U}(y)-C>0$

As $d^2\bar{U}/dy^2<0$ that means $-(-d^2\bar{U}/dy^2)<0$, then $-(d\zeta/dy)<0$ and that is $d\zeta/dy>0$, the gradient of vorticity is increase along y axi, and U(y) is monotony increase along y, wind of $\bar{U}(y)$ is in antivorticity state, that means wind field has negative vorticity.

That is the wind field offer negative vorticity to the perturbation along y axi. But the gradient of vorticity is also increase along y axis. So the environment is not benefit to the disturbance. And the disturbance is stable.

Conclusion

The necessary condition of barotropic instability is that at some value of y_k, the vorticity must change sign. In our research and the basic flow must also satisfy $(d^2U/dy^2)\times(U-C)\leq 0$ and the wave speed relative to the mean flow. Then the disturbance will develop.

Reference

Kuo, J. L., 1951 Dynamic aspects of general circulation and stability of zonal flow. Tellus, 3. 268- 284.

Tollmien , W., 1935 Ein Allgemeines Kriterum der instabilitat laminerar geschwindigkeitseverteilungen,Nachr,Ges.Wiss.Gottingen,Math.Phys. lasse,50,1935.