An explicit finite difference model for simulating weakly nonlinear and weakly dispersive waves over slowly varying water depth

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Abstract

In this paper, a modified leap-frog finite difference (FD) scheme is developed to solve Nonlinear Shallow Water Equations (NSWE). By adjusting the FD mesh system and modifying the leap-frog algorithm, the numerical dispersion is manipulated to mimic the physical frequency dispersion for water waves in the regime of intermediate water depth. The resulting numerical scheme is suitable for weakly dispersive waves over a slowly varying water depth. A numerical experiment has been carried out to demonstrate that the results of the new numerical scheme agree well with those obtained by directly solving Boussinesq-type models for shoaling and refraction over a slowly varying bathymetry. Most importantly, the present algorithm is much more computational efficient than existing Boussinesq-type models, making it an excellent alternative tool for simulating tsunami waves when the frequency dispersion needs to be considered.

Key words: Numerical methods; Shallow water wave equations; Boussinesq equations; Frequency dispersion.

1. Introduction

In the past several decades, several numerical models have been developed for calculating transoceanic tsunami propagation. Most of these numerical models are based on the Shallow Water Equations (SWE), which is justified because the wavelength of a typical tsunami is usually much larger than water depth so that the frequency dispersion can be ignored. Because of the necessity of quickly producing numerical results for tsunami early warning system, SWE-based models usually adopt explicit finite difference schemes (e.g., Liu et al., 1995). On the other hand, tsunami propagation models based on Boussinesq-type (BT) equations are capable of considering frequency dispersive effects from shallow to intermediate water. However, because of the appearance of higher order terms associated with the frequency dispersion, the algorithms for BT models call for finer spatial and temporal resolution and higher order numerical algorithms. Furthermore, since implicit methods are employed in BT models significant more computational resources and computational times are required.

Imamura et al. (1988) (hereafter IM88) presented a FD model for the simulation of transoceanic tsunamis, which solves the Linear Shallow Water Equations (LSWE) using the explicit leap-frog scheme. The frequency dispersion terms neglected in the LSWE are taken into account by utilizing the numerical dispersion inherent in the leap-frog FD scheme. This is done by choosing the grid size and time step according to a specified criterion. However, the frequency dispersion effects obliquely to the principle axes of the computational domain were not properly represented in the original algorithm. Cho (1995) (hereafter CH95) improved upon IM88's numerical algorithm so that frequency dispersion effects in all directions of tsunami propagation are correctly included. Consequently, the numerical algorithm actually produces numerical results satisfying the traditional Boussinesq equations in a constant water depth.

When the frequency dispersion is important in simulating tsunami propagation over a varying water depth, the frequency dispersion effects need to be carefully considered at every grid point in the entire computational domain. Thus, following the framework of IM88 and CH95, the grid size needs to be locally adjusted according to the time step and the local water depth, which makes the implementation of these algorithms somewhat difficult. Yoon (2002) developed a new FD scheme that satisfies the local frequency dispersion requirement for a varying water depth while a uniform grid system is still employed. In Yoon's method a hidden moving grid system determined locally from the condition suggested by IM88 is introduced. The physical variables associated with the hidden moving grid system are obtained by interpolating the variables assigned on the actual uniform grid points. Yoon (2002) demonstrated that this scheme provides significant improvements on the frequency dispersion effects compared to those of IM88 and CH95

More recently, Yoon et al. (2007) developed another scheme in which the linearized BT equations are resolved with an explicit FD method. The resulting numerical dispersion is again used to improve the physical frequency dispersion. We note that in Yoon et al. (2007) the BT equations were combined into a wave equation in terms of the free surface displacement. A finite element (FE) version of this model is also available in Yoon et al. (2008). Since their models are developed based on linearized Boussinesq equations and the variation of water depth is also taken into account, they show good performance for dispersive waves over variable water depth and the computational efficiency is very high. However, their models are only applicable for linear waves. As tsunamis shoal onto a continental shelf, nonlinearity gradually plays a significant role in the transformation. Linear model is no longer useful. Moreover, since only free surface elevations are solved from the models by Yoon et al. (2007, 2008) and the velocity field must be solved separately, the estimation of the computational efficiency is not necessarily conservative.

In this paper, a modified leap-frog FD method is proposed to solve the Nonlinear Shallow Water Equations (NSWE) over a slowly varying water depth. Adopting the idea of a moving hidden grid system suggested by Yoon (2002), the numerical dispersion is manipulated to recover the physical frequency dispersion neglected in the NSWE. In the new algorithm, the numerical errors generated by discretizing the nonlinear terms are eliminated by adding correction terms to the original leap-frog FD scheme so that the numerical dispersion can still be used to recover the physical dispersion in the classical Boussinesq equations. Several numerical tests are carried out to show that the proposed algorithm can accurately simulate evolution of weakly nonlinear waves over a constant or slowly varying water depth. Most importantly, the proposed model still adopts explicit schemes and has a much higher computational efficiency than existing BT models.

2. Governing Equations

The depth-integrated Boussinesq equations over a varying water depth can be written in the following form (CH95):

$$\frac{\partial \eta}{\partial t}+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=0\,, \tag{1}$$

$$\begin{aligned}\frac{\partial P}{\partial t}+\epsilon\left[\frac{\partial}{\partial x}\left(\frac{P^2}{H}\right)+\frac{\partial}{\partial y}\left(\frac{PQ}{H}\right)\right]+H\frac{\partial \eta}{\partial x}=-\mu^2\frac{h^3}{3}\left[\frac{\partial^3\eta}{\partial x^3}+\frac{\partial^3\eta}{\partial x\partial y^2}\right]\\-\mu^2\frac{h^2}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial \eta}{\partial y}\right)+\frac{\partial h}{\partial x}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)\right]+O(\mu^4,\epsilon\mu^2),\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial Q}{\partial t}+\epsilon\left[\frac{\partial}{\partial x}\left(\frac{PQ}{H}\right)+\frac{\partial}{\partial y}\left(\frac{Q^2}{H}\right)\right]+H\frac{\partial \eta}{\partial y}=-\mu^2\frac{h^3}{3}\left[\frac{\partial^3\eta}{\partial x^2\partial y}+\frac{\partial^3\eta}{\partial y^3}\right]\\-\mu^2\frac{h^2}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial \eta}{\partial y}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)\right]+O(\mu^4,\epsilon\mu^2),\end{aligned} \tag{3}$$

where P and Q are the volume flux in the x and y direction, respectively and η is the free surface displacement. Two small parameters, ϵ and μ,

$$\epsilon=\frac{a_0}{h_0} \quad \text{and} \quad \mu=\frac{h_0}{l_0}, \tag{4}$$

denote the nonlinearity and frequency dispersion, respectively. In the Boussinesq approximation both effects of nonlinearity and frequency dispersion are equally weak. The right-hand side of the momentum equations, (2) and (3), represents the frequency dispersion, in which the terms associated with $\mu^2h^2/2$ reflect the effects of varying water depth.

When the frequency dispersion effects are ignored ($\mu=0$), the right-hand side of the above momentum equations, (2) and (3), are neglected. The resulting equations are the NSWE.

$$\frac{\partial \eta}{\partial t}+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=0, \tag{5}$$

$$\frac{\partial P}{\partial t}+\epsilon\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial y}\right)+H\frac{\partial \eta}{\partial x}=0, \tag{6}$$

$$\frac{\partial Q}{\partial t}+\epsilon\left(\frac{\partial W}{\partial x}+\frac{\partial V}{\partial y}\right)+H\frac{\partial \eta}{\partial y}=0\,, \tag{7}$$

where $U=P^2/H, V=Q^2/H$ and $W=(PQ)/H$. For most of transoceanic tsunamis, wave amplitudes are small compared to water depth and the nonlinearity, ϵ, can be neglected. Dropping the nonlinear terms (6) and (7), the NSWE is reduced to the LSWE.

3. Numerical Scheme

IM88 and CH95 introduced an algorithm using the modified leap-frog FD scheme to discretize the LSWE so that inherent numerical dispersion errors are manipulated to mimic the physical frequency dispersion effects for tsunamis propagating over a constant water depth. To achieve this, the following condition involving the grid size (Δx), time step (Δt) and water depth needs to be satisfied:

$$\Delta x = \sqrt{4h^2 + gh(\Delta t)^2} \qquad (8)$$

Yoon (2002) extended this approach to include the effects of slowly varying water depth. Clearly, from (8) the grid size has to be varying according to local water depth, making the direct implementation problematic. Yoon proposed an elegant way to delineate this problem. Besides a fixed uniform computational grid system in which the governing equations are discretized, a local (moving) hidden grid system is then introduced whose grid size is determined by the local water depth using the condition (8). Therefore, the local numerical dispersion can be adjusted to locally mimic physical dispersion.

In our present model, Yoon's approach is extended for solving the NSWE over slowly varying water depth are given as follows:

$$\frac{\eta_{i,j}^{n+1/2} - \eta_{i,j}^{n-1/2}}{t} + \frac{P_{*F,j}^{n} - P_{*B,j}^{n}}{x_*} + \frac{Q_{*U,i}^{n} - Q_{*L,i}^{n}}{y_*} = 0, \qquad (9)$$

$$\begin{aligned}
&\frac{P_{i+1/2,j}^{n+1} - P_{i+1/2,j}^{n}}{t} + \grave{o}\left(\frac{U_{i+1/2,j}^{n} - U_{i-1/2,j}^{n}}{x} + \frac{W_{i,j+1/2}^{n} - W_{i,j-1/2}^{n}}{y}\right) + H_{i+1/2,j}\frac{\eta_{F,j}^{n+1/2} - \eta_{B,j}^{n+1/2}}{x_*} \\
&+ \frac{\grave{o}}{2x}\left[\alpha(U_{i+3/2,j}^{n} - U_{i+1/2,j}^{n}) - (\alpha-1)(U_{i+1/2,j}^{n} - U_{i-1/2,j}^{n}) - (U_{i+1/2,j}^{n-1} - U_{i-1/2,j}^{n-1})\right] \\
&+ \frac{\grave{o}}{2y}\left[\alpha(W_{i+1,j+1/2}^{n} - W_{i+1,j-1/2}^{n}) - (\alpha-1)(W_{i,j+1/2}^{n} - W_{i,j-1/2}^{n}) - (W_{i,j+1/2}^{n-1} - W_{i,j-1/2}^{n-1})\right] \\
&+ \frac{\alpha^2 h_{i+1/2,j}}{12x}\left[(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i+1,j}^{n+1/2} + \eta_{i+1,j-1}^{n+1/2}) - (\eta_{i,j+1}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i,j-1}^{n+1/2})\right] = 0
\end{aligned} \qquad (10)$$

$$\begin{aligned}
&\frac{Q_{i,j+1/2}^{n+1} - Q_{i,j+1/2}^{n}}{t} + \grave{o}\left(\frac{W_{i+1/2,j}^{n} - W_{i-1/2,j}^{n}}{x} + \frac{V_{i,j+1/2}^{n} - V_{i,j-1/2}^{n}}{y}\right) + H_{i,j+1/2}\frac{\eta_{U,i}^{n+1/2} - \eta_{L,i}^{n+1/2}}{y_*} \\
&+ \frac{\grave{o}}{2y}\left[\alpha(V_{i,j+3/2}^{n} - V_{i,j+1/2}^{n}) - (\alpha-1)(V_{i,j+1/2}^{n} - V_{i,j-1/2}^{n}) - (V_{i,j+1/2}^{n-1} - V_{i,j-1/2}^{n-1})\right] \\
&+ \frac{\grave{o}}{2x}\left[\alpha(W_{i+1/2,j+1}^{n} - W_{i-1/2,j+1}^{n}) - (\alpha-1)(W_{i+1/2,j}^{n} - W_{i-1/2,j}^{n}) - (W_{i+1/2,j}^{n-1} - W_{i-1/2,j}^{n-1})\right] \\
&+ \frac{\alpha^2 h_{i,j+1/2}}{12y}\left[(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i,j+1}^{n+1/2} + \eta_{i-1,j+1}^{n+1/2}) - (\eta_{i+1,j}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i-1,j}^{n+1/2})\right] = 0
\end{aligned} \qquad (11)$$

where the subscripts F, B, U, and L represent forward, backward, upper and lower grid points on the hidden grid system associated with the grid point (i, j) on the fixed uniform grid system for the continuity equation (9), $(i+1/2, j)$ for the momentum equation (10) in x-direction and $(i, j+1/2)$ for the momentum equation (11) in y-direction, respectively. We reiterate that Δx and Δy are the size of fixed uniform grids in x- and y- direction and $\Delta x = \Delta y$. Δx^* denotes the size of hidden grids and determined by (8) and $\alpha = \Delta x_* / \Delta x$.

In the following section, a numerical test is performed to investigate the validity and application range of the proposed numerical algorithm.

A Numerical Example

Yoon et al. (2007) presented numerical simulations of the transformation of tsunami waves over a submerged shoal based on their own model and a Boussinesq model, FUNWAVE (Wei and Kirby, 1995). The same numerical experiment is carried out here and the present results are compared with those of Yoon et al.

The numerical domain is 1500 km long and 500 km wide with a constant water depth of 1500 m. The submerged circular shoal is located at (x_0, y_0) = (500km, 250km) as shown in Figure 1 and the water depth over the shoal is given as

$$h(r) = 1500, \quad r \ge R_1; \; h(r) = 1500(r^2/R_1^2); \; h(r) = 500, \quad r \le R_2 \qquad (12)$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and $R_1 = 150km$ and $R_2 = 86.6km$, which yields a slope roughly of 1/64. Four gauges are deployed to record the time histories of water surface elevations. Gage 1 and 3 are located on the slope where water depth is 1000 m. Gage 2 is located at the center of shelf top and Gage 4 is located at (750km, 0.0km). At both ends of the numerical domain, sponge layer is deployed to avoid wave reflection. Vertical wall is implemented along lateral boundaries.

Along $x = 0.0$, the initial condition is given as

$$\eta(x, t=0) = a_0 e^{-x^2/x_0^2}, \; P(x, t=0) = 0.0, \; Q(x, t=0) = 0.0 \qquad (13)$$

where a_0 = 2.0m and x_0=7500m. This is a Gaussian hump uniform in y-direction. The characteristic length scale of this Gaussian profile is about 15km, which yields $\mu = 0.1$. Considering the extremely small ratio of the amplitude to water depth, $\grave{o} = 0.0013$, the nonlinearity is neglected during simulation.

In the numerical simulation, the grid size and time step are selected to be the same as those used by Yoon et al. (2007), which are $\Delta x = 2000m$ and $\Delta t = 4s$. The numerical results shows a good agreement with those by Yoon et al. at all four gauge locations (see Figures 2 and 3). Although the present numerical results become slightly more dispersive than those obtained by Yoon et al. (2007) as waves propagate further away from the source, the first several waves, which are crucially important in determining tsunami runup and inundation, match perfectly with those of the Boussinesq model (FUNWAVE) and Yoon et al.'s (2007).

Concluding remarks

In this paper, a modified leap-frog finite difference scheme is developed to solve nonlinear shallow water equations. By properly adjusting the grid size in terms of local water depth and time step size, the numerical dispersion errors inherent in the explicit leap-frog finite difference scheme is manipulated to recover the physical frequency dispersion. The new model is able to simulate weakly nonlinear and weakly dispersive waves propagating over a slowly varying water depth.

The numerical example presented in this paper shows that, for both long wave propagation over a long distance and wave shoaling onto a mild slope, numerical results of the present model are in very good agreement with those from Boussinesq models. This greatly extends the application range of traditional Nonlinear Shallow Water Equations from the shallow water to intermediate water depth. The biggest advantage of the present model is its efficiency. Simple examples of solitary wave propagating over a constant depth or a uniform slope, the CPU times required by the Boussinesq equation models are two to three orders of magnitudes of that required by the present model. Apparently, using the leap-frog explicit scheme, the present model can be used to solve a large computational domain.

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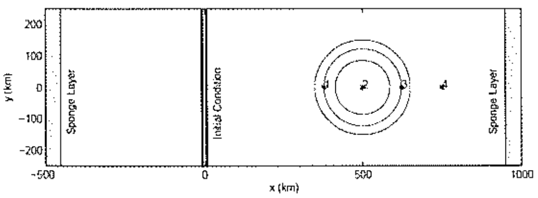

Figure 1 A sketch of the computational domain

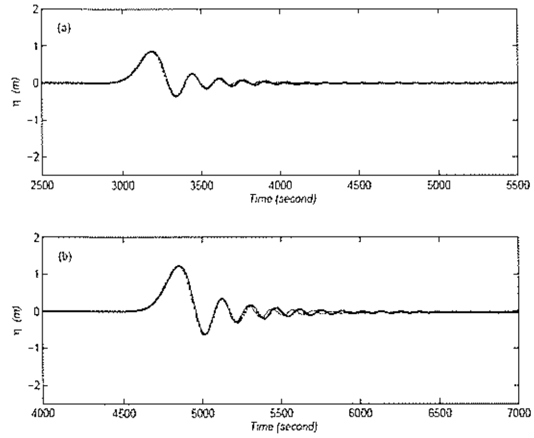

Figure2 Comparisons of numerical results obtained from the present model and those by Yoon et al. (2007).

The dotted lines denote the numerical results from the present model; the solid lines are the numerical results of FUNWAVE (Yoon et al., 2007) and the dash-dot lines are Yoon's results (2007). And the sub-plot (a) and (b) show the comparisons at Gage 1 and 2, respectively.

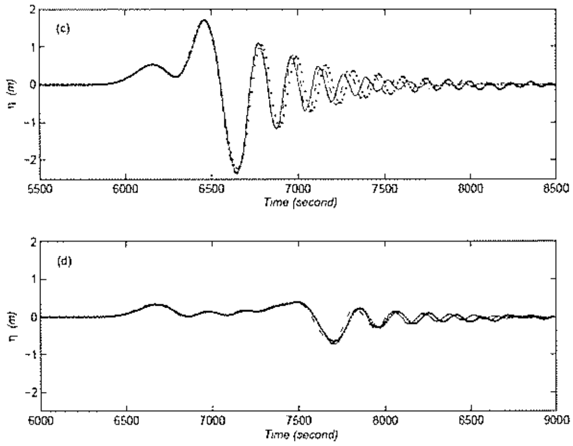

Figure 3 Comparisons of numerical results obtained from the present model and those by Yoon et al. (2007) for Gage 3 (c) and Gage 4 (d) , respectively.